

# The Inversion Relation and the Dilute $A_{3, 4, 6}$ Eigenspectrum<sup>1</sup>

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On the basis of the result obtained by applying Baxter's exact perturbative approach to the dilute  $A_3$  model to give the  $E_8$  mass spectrum, the dilute  $A_L$  inversion relation was used to predict the eigenspectra in the  $L=4$  and  $L=6$  cases (corresponding to  $E_7$  and  $E_6$  respectively). In calculating the next-to-leading term in the correlation lengths, or equivalently masses, the inversion relation condition gives a surprisingly simple result in all three cases, and for all masses.

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**KEY WORDS:** Ising model in a field; dilute A model; integrable quantum field theory; mass spectrum.

## 1. INTRODUCTION

One model in statistical mechanics which attracts perennial interest is the two-dimensional Ising model in a magnetic field. The integrable quantum field theory which describes the (massive) scaling limit of this model is the  $(1, 2)$ -perturbation of  $c = \frac{1}{2}$  conformal field theory due to Zamolodchikov<sup>(1)</sup> which has  $E_8$  symmetry. Among the hierarchy of models, in their four regimes, which form the dilute  $A_L$  model,<sup>(2)</sup> is a realisation of this Ising model.

The  $A_L$  model is an  $L$  state interaction-round-a-face model<sup>(3)</sup> whose adjacency diagram is the Dynkin diagram of  $A_L$  with the additional

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possibility of a state being adjacent to itself on the lattice. In regime 2, the central charge is

$$c = 1 - \frac{6}{L(L+1)}, \quad (1)$$

and for  $L$  odd the  $Z_2$  symmetry is broken away from criticality. As well as general calculations for dilute  $A_L$  with  $L$  odd, various results have been obtained which demonstrate the Ising critical exponents<sup>(4,5)</sup> and hidden  $E_8$  structures<sup>(6-13)</sup> in the dilute  $A_3$  model. Further, for  $L=4$  and  $L=6$ , the central charge of the  $E_7$  and  $E_6$  field theories<sup>(14)</sup> are recovered from (1). While no complete calculation of order parameters for dilute  $A_L$  with  $L$  even has yet been carried out, there is a growing literature concerning the hidden E-type structures.<sup>(6,15,16)</sup>

The motivation for the result presented here is a recent paper<sup>(17)</sup> in which are given arguments for the higher-order terms in the scaling forms for the Ising free energy and mass spectrum, and numerical estimates for some of the corresponding amplitudes and universal amplitude ratios. In Section 2 we review our previous results for the eigenspectrum of the dilute  $A_L$  model for  $L=3, 4, 6$ , or equivalently the mass spectrum for  $E_8, E_7$  and  $E_6$ , with particular reference to the way the inversion relation of the solvable model is expressed through it. In Section 3 we calculate the first correction term for all three cases,  $E_{6,7,8}$  and, as a consequence of this property of the lattice model realisation, we are able to give the coefficient very simply.

## 2. THE MASS SPECTRA

The eigenvalues of the row-to-row transfer matrix of the dilute  $A$  models are<sup>(7)</sup>

$$\begin{aligned} \Lambda(u) = & \omega \left[ \frac{\vartheta_1(2\lambda - u) \vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)} \right]^N \prod_{j=1}^N \frac{\vartheta_1(u - u_j + \lambda)}{\vartheta_1(u - u_j - \lambda)} \\ & + \left[ \frac{\vartheta_1(u) \vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)} \right]^N \prod_{j=1}^N \frac{\vartheta_1(u - u_j) \vartheta_1(u - u_j - 3\lambda)}{\vartheta_1(u - u_j - \lambda) \vartheta_1(u - u_j - 2\lambda)} \\ & + \omega^{-1} \left[ \frac{\vartheta_1(u) \vartheta_1(\lambda - u)}{\vartheta_1(2\lambda) \vartheta_1(3\lambda)} \right]^N \prod_{j=1}^N \frac{\vartheta_1(u - u_j - 4\lambda)}{\vartheta_1(u - u_j - 2\lambda)} \end{aligned} \quad (2)$$

where the  $N$  roots  $u_j$  are given by the Bethe equations

$$\omega \left[ \frac{\vartheta_1(\lambda - u_j)}{\vartheta_1(\lambda + u_j)} \right]^N = - \prod_{k=1}^N \frac{\vartheta_1(u_j - u_k - 2\lambda) \vartheta_1(u_j - u_k + \lambda)}{\vartheta_1(u_j - u_k + 2\lambda) \vartheta_1(u_j - u_k - \lambda)} \quad (3)$$

and  $\omega = \exp(i\pi\ell/(L + 1))$  for  $\ell = 1, \dots, L$ . Here the (standard) elliptic functions have nome  $p = e^{-\varepsilon}$  and for the regime of interest, the spectral parameter obeys  $0 < u < 3\lambda$ , where the crossing parameter is  $\lambda = \pi s/r$  and in terms of  $L$ ,  $s = L + 2$  and  $r = 4(L + 1)$ .

Based on numerical data for the string structure and positions of the Bethe ansatz roots for the dilute  $A_3$  model,<sup>(7,8)</sup> which indicate that there are eight excitations of thermodynamic significance, recurrence relations which enable the eigenvalue expressions to be found were solved.<sup>(10,11)</sup> The technique used is an exact perturbative approach, about the ordered limit (with  $N$  large), first developed by Baxter for the eight-vertex model,<sup>(18,3)</sup> and also applied to the cyclic solid-on-solid model.<sup>(19)</sup> In terms of the conjugate variables  $w = e^{-2\pi u/\varepsilon}$  and  $x = e^{-\pi^2/r\varepsilon}$ , the ordered limit is  $x \rightarrow 0$  ( $w$  fixed). For the  $L = 3$  case, this corresponds to the strong-field limit, while for the  $L$  even cases, this is the low-temperature situation. The elliptic functions in the conjugate modulus form of (2) are those defined below in (7), with nome  $x^{2r}$ .

The quantities actually determined are the excitations

$$r_j(w) = \lim_{N \rightarrow \infty} \frac{A_j(w)}{A_0(w)} \tag{4}$$

in terms of which the correlation lengths are  $\xi_j^{-1} = -\log r_j$ . Because of the inversion and crossing relations obeyed by the model's Boltzmann weights,<sup>(4)</sup> the excitations must satisfy the inversion relation:

$$r_j(w) r_j(x^{6s}w) = 1, \tag{5}$$

while a consequence of (2) is the stronger relation, which implies the former,

$$r_j(w) r_j(x^{4s}w) = r_j(x^{2s}w). \tag{6}$$

In terms of the (conjugate modulus) elliptic functions

$$E(z, q) = \prod_{n=1}^{\infty} (1 - q^{n-1}z)(1 - q^n/z)(1 - q^n) = E(q/z, q) = -zE(z^{-1}, q), \tag{7}$$

the expression obtained<sup>(11)</sup> is

$$r_j(w) = w^{n(a)} \prod_a \frac{E(-x^a/w, x^{60}) E(-x^{30-a}/w, x^{60})}{E(-x^a w, x^{60}) E(-x^{30-a} w, x^{60})}. \tag{8}$$

Apparent in this expression is the band structure of the eigenspectrum, labelled by powers of  $w$ , the power given by the number of integers  $n(a)$  appearing in the product. The values  $a$  takes, arising from the calculation

**Table I. The Integers  $a$  which Appear in the Eigenvalue Expression (8) of the Dilute  $A_3$  Model**

$j$	$n(a)$	$a$
1	2	1, 11
2	2	7, 13
3	3	2, 10, 12
4	3	6, 10, 14
5	4	3, 9, 11, 13
6	4	6, 8, 12, 14
7	5	4, 8, 10, 12, 14
8	6	5, 7, 9, 11, 13, 15

described above, are those given in Table I. Transforming to the original variables, appropriate for the critical limit, the leading term is

$$m_j = \xi_j^{-1} \sim 8p^{8/15} \sum_a \sin \frac{a\pi}{30} \quad \text{as } p \rightarrow 0, \quad (9)$$

which gives the mass ratios of the  $E_8$  field theory.<sup>(1)</sup>

There is no explicit  $L$  dependence in the integers  $a$  which could be generalised to other members of the dilute  $A$  hierarchy in an immediate way. However, they have appeared in connection with  $E_8$  in various contexts, for example ref. 20, and with various interpretations. Most significantly, they occur in affine Toda theory, where they appear in the  $S$ -matrix for scattering from the particle labelled  $m_1$ .<sup>(21)</sup> In the context of the dilute  $A_3$  model, McCoy and Orrick<sup>(9)</sup> observed them in work related to ref. 7, and hence to the same Bethe ansatz root string structure used in the calculation of (8). Suzuki<sup>(12)</sup> has used the quantum transfer matrix (QTM) approach to recover the  $E_8$  Bethe ansatz equation from dilute  $A_3$  *without* any assumption of particular string structure, and has remarked the occurrence of the same integers in the zeroes of the fusion QTMs. This suggests that for the cases  $L=4, 6$  an expression analogous to (8) should describe the eigenspectra, since the integers of Table I have  $E_6$  and  $E_7$  counterparts.

In expression (8) “30” plays a distinguished role in relationship to the new nome  $x^{12s}$ , where  $s=L+2$ . On the other hand, we see that in (9) it enters as the dual Coxeter number  $g=30$  of  $E_8$ . Moreover, the universal amplitude<sup>(22, 23)</sup>

$$f_s \xi_1^2 = 0.061728\dots \quad (10)$$

is obtained,<sup>(10)</sup> and this relies on the power of  $p$  in the correlation length  $\xi_j$  being appropriately related to that of the singular part of the free energy of the dilute  $A_L$  model,<sup>(4)</sup>

$$f_s \sim p^{r/3s} \quad \text{as } p \rightarrow 0. \tag{11}$$

Although (8) would obey (5) for any integer  $a$ , the stronger inversion relation (6) is satisfied because each integer  $a$  in Table I occurs together with  $a + 2s = a + (g/3)$ , or equivalently, from properties of the elliptic function (7), with  $4s - a = (2g/3) - a$ .

Gathering together these observations, it was proposed<sup>(15)</sup> that (8) is a special case of the expression

$$r_j(w) = w^{n(a)} \prod_a \frac{E(-x^{6sa/g}/w, x^{12s}) E(-x^{(6s(g-a))/g}/w, x^{12s})}{E(-x^{6sa/g}w, x^{12s}) E(-x^{(6s(g-a))/g}w, x^{12s})}. \tag{12}$$

For the dilute  $A_4$  model, which is related to  $E_7$ ,  $g = 18$ , while for the  $A_6$  model the appropriate Coxeter number is that of  $E_6$ ,  $g = 12$ .

Looking among the  $E_{6,7}$  analogues (in refs. 20 and 21) of the integers in Table I, knowing that the integers  $a$  appearing in (12) must be such that (6) is obeyed, we were led to consider those in Tables II and III. The mass ratios of  $E_7$  and  $E_6$  respectively are correctly given by the leading term when (12) is expressed in the original nome,

$$\xi_j^{-1} = m_j \sim 8p^{r/6s} \sum_a \sin \frac{a\pi}{g} \quad \text{as } p \rightarrow 0. \tag{13}$$

The integers in Tables II and III correspond to scattering from particle  $m_2$  in the  $S$ -matrix of ref. 21, unlike those in Table I which we noted before

**Table II. The Integers to Appear in (12) in the  $L = 4$ , or Equivalently  $E_7$  Case**

$j$	$n(a)$	$a$
1	1	6
2	2	1, 7
3	2	4, 8
4	2	5, 7
5	3	2, 6, 8
6	3	4, 6, 8
7	4	3, 5, 7, 9

**Table III. The Integers to Appear in (12) in the  $L = 6$  Case**

$j$	$n(a)$	$a$
1, $\bar{1}$	1	4
2	2	1, 5
3, $\bar{3}$	2	3, 5
4	3	2, 4, 6

corresponded to  $m_1$ . This appears to be related to the concrete connection drawn in ref. 24 between Toda theory related to affine Lie algebras and integrable perturbations of conformal field theory. The additional node on the Dynkin diagram in the affine case connects to the node called  $m_1$  (resp.  $m_2, m_2$ ) in the field theory labelling of the  $E_8$  (resp.  $E_7, E_6$ ) diagram, thus distinguishing this node. We should also remark that Suzuki<sup>(16)</sup> has recently obtained these integers in analogous work to ref. 12 for  $L = 4, 6$ .

The conjecture (12) has now been confirmed<sup>(26)</sup> by using the string structure for the Bethe ansatz roots for dilute  $A_4$ <sup>(25)</sup> to perform the same type of calculation that led to (8), for the  $L = 4$  case.

### 3. HIGHER-ORDER TERMS

Recently, Caselle and Hasenbusch<sup>(17)</sup> have presented arguments based on the renormalization group approach to give higher-order terms appearing in the scaling form of the free energy and mass spectrum of the critical two-dimensional Ising model in a magnetic field, and have obtained numerical estimates for the amplitudes of some of these subleading corrections. Since the dilute  $A_3$  model is from the same universality class, it seems pertinent to examine the higher-order terms in the mass spectrum (12) to establish both their order and the coefficients.

The mass spectrum of the Ising model is given in the form (our notation for coefficients)

$$m_j^2(h) = \mathcal{A}_{m_j}^2 h^{16/15} (1 + \mathcal{B}_{m_j} h^{16/15} + \mathcal{C}_{m_j} h^{22/15} + \mathcal{D}_{m_j} h^{30/15} + \mathcal{E}_{m_j} h^{32/15} + \dots), \quad (14)$$

which includes contributions attributed to both primary (relevant) and secondary (irrelevant) operators.<sup>(17)</sup>

Now consider the dilute A expression (12) expressed in terms of the original nome,  $p$ , which plays the role of the magnetic field  $h$  for  $L=3$  (while for  $L$  even  $p=0$  corresponds to critical temperature):

$$m_j = 2 \sum_a \log \frac{\mathcal{G}_4(a\pi/2g + \pi/4, p^{r/6s})}{\mathcal{G}_4(a\pi/2g - \pi/4, p^{r/6s})}. \tag{15}$$

Using the elliptic theta function identity<sup>(27)</sup>

$$\log \frac{\mathcal{G}_4(\alpha + \beta, q)}{\mathcal{G}_4(\alpha - \beta, q)} = 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 - q^{2n}} \sin(2n\alpha) \sin(2n\beta) \tag{16}$$

the masses may be written as

$$m_j = 8 \sum_a \sum_{n=1}^{\infty} \frac{1}{n} \frac{(p^{r/6s})^n}{1 - (p^{r/6s})^{2n}} \sin\left(\frac{na\pi}{g}\right) \sin\left(\frac{n\pi}{2}\right) \tag{17}$$

from which it can be seen that only odd-integer powers of  $p^{r/6s}$  will occur, i.e., odd powers of  $p^{8/15}$  for the  $L=3$  case.

Expanding (17) in powers of  $p^{r/6s}$  yields

$$m_j = 8p^{r/6s} \left\{ \sum_a \sin\left(\frac{a\pi}{g}\right) + \sum_a \left[ \sin\left(\frac{a\pi}{g}\right) - \frac{1}{3} \sin\left(\frac{3a\pi}{g}\right) \right] (p^{r/6s})^2 + \dots \right\}, \tag{18}$$

which appears unwieldy, particularly since each set of possible  $a$ 's may contain a different number of values. However, the property of the sets of integers appearing in Tables I–III which caused (12) to obey the inversion relation (6), namely that  $a$  occurs together with  $a + (g/3)$  (or  $(2g/3) - a$ ), means that

$$\sum_a \sin\left(\frac{3a\pi}{g}\right) = 0 \tag{19}$$

so that we obtain

$$m_j = 8p^{r/6s} \left\{ \sum_a \sin \frac{a\pi}{g} \right\} \left\{ 1 + (p^{r/6s})^2 + \mathcal{O}((p^{r/6s})^4) \right\}. \tag{20}$$

We thus see that the next-to-leading term has the same simple coefficient for each of the cases  $E_{6,7,8}$  as a consequence of the inversion relation of the dilute  $A_L$  model. In particular, the amplitude is constant for each of the masses. This is in contrast with the numerical estimates of the amplitudes  $\mathcal{B}_{m_j}$  in (14) for the Ising model, which are clearly mass dependent.<sup>(17)</sup> Although the dilute  $A_3$  model,  $E_8$  and the magnetic Ising model all

share the same critical behaviour in the scaling limit, it does not follow that their universality extends to the sub-leading correction terms. For this reason it may well not be possible to predict the amplitudes of the sub-leading corrections for the Ising model.

As a final remark we note that the next term in the expansion of (17) has coefficient

$$\sum_a \left[ \sin\left(\frac{a\pi}{g}\right) + \frac{1}{5} \sin\left(\frac{5a\pi}{g}\right) \right] \quad (21)$$

and for the integers  $a$  in Table I which apply to the Ising case (but not for those in Tables II and III) the second term sums again to zero. Thus the  $L=3$  mass spectrum is given by

$$m_j = 8p^{8/15} \left\{ \sum_a \sin\frac{a\pi}{30} \right\} \{1 + p^{16/15} + p^{32/15} + \mathcal{O}(p^{48/15})\}. \quad (22)$$

However, there is no simple pattern to the higher-order coefficients. The  $\mathcal{O}(p^{48/15})$  term in (22) is mass dependent. For the first two mass ratios we find the leading correction to be

$$\frac{m_2}{m_1} = 2 \cos \frac{\pi}{5} \left( 1 + \frac{\sqrt{5}}{7} p^{48/15} \right) \simeq 1.618(1 + 0.319p^{48/15}) \quad (23)$$

$$\frac{m_3}{m_1} = 2 \cos \frac{\pi}{30} \left( 1 + \frac{\sin 2\pi/15}{7 \cos \pi/30} p^{48/15} \right) \simeq 1.989(1 + 0.058p^{48/15}). \quad (24)$$

The convergence to Zamolodchikov's predicted mass ratios in the dilute  $A_3$  model is thus more rapid than that observed for the Ising model.

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## REFERENCES

1. A. B. Zamolodchikov, *Adv. Stud. Pure Math.* **19**:641–674 (1989); *Int. J. Mod. Phys. A* **4**:4235–4248 (1989).
2. S. O. Warnaar, B. Nienhuis, and K. A. Seaton, *Phys. Rev. Lett.* **69**:710–712 (1992); *Int. J. Mod. Phys. B* **7**:3727–3736 (1993).



3. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
4. S. O. Warnaar, P. A. Pearce, K. A. Seaton, and B. Nienhuis, *J. Stat. Phys.* **74**:469–531 (1994).
5. M. T. Batchelor, V. Fridkin, and Y. K. Zhou, *J. Phys. A* **29**:L61–67 (1996).
6. S. O. Warnaar and P. A. Pearce, *J. Phys. A: Math. Gen.* **27**:L891–L897 (1994); *Int. Journ. Mod. Phys. A* **11**:291–311 (1996).
7. V. V. Bazhanov, B. Nienhuis, and S. O. Warnaar, *Phys. Lett. B* **322**:198–206 (1994).
8. U. Grimm and B. Nienhuis, *Phys. Rev. E* **55**:5011–5025 (1997).
9. B. M. McCoy and W. P. Orrick, *Phys. Lett. A* **230**:24–32 (1997).
10. M. T. Batchelor and K. A. Seaton, *J. Phys. A* **30**:L479–484 (1997).
11. M. T. Batchelor and K. A. Seaton, *Nucl. Phys. B* **520**:697–744 (1998).
12. J. Suzuki, *Nucl. Phys. B* **528**:683–700 (1998).
13. Y. Hara, M. Jimbo, H. Konno, S. Odake, and J. Shiraishi, *J. Math. Phys.* **40**:3791–3826 (1999).
14. V. A. Fateev and A. B. Zamolodchikov, *Int. Journ. Mod. Phys. A* **5**:1025–1048 (1990).
15. M. T. Batchelor and K. A. Seaton, *Eur. Phys. J. B* **5**:719–725 (1998); K. A. Seaton and M. T. Batchelor, *Group22: Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics*, S. P. Corney, R. Delbourgo, and P. D. Jarvis, eds. (International Press, Cambridge, MA, 1999), pp. 274–278.
16. J. Suzuki, in *Physical Combinatorics*, M. Kashiwara and T. Miwa, eds. (Birkhäuser Boston, Cambridge, MA, 2000), pp. 217–247.
17. M. Caselle and M. Hasenbusch, *Nucl. Phys. B* **579**:667–703 (2000).
18. R. J. Baxter, *Ann. Phys. (N. Y.)* **70**:193–228 (1972).
19. P. A. Pearce and M. T. Batchelor, *J. Stat. Phys.* **60**:77–135 (1990).
20. B. Kostant, *Proc. Natl. Acad. Sci. USA* **81**:5275–5277 (1984).
21. H. W. Braden, E. Corrigan, P. E. Dorey, and R. Sasaki, *Nucl. Phys. B* **338**:689–746 (1990).
22. V. A. Fateev, *Phys. Lett. B* **324**:45–51 (1994).
23. G. Delfino and G. Mussardo, *Nucl. Phys. B* **455**:724–758 (1995).
24. T. J. Hollowood and P. Mansfield, *Phys. Lett. B* **226**:73–79 (1989).
25. U. Grimm and B. Nienhuis, private communication.
26. K. A. Seaton and M. T. Batchelor, in preparation.
27. A. Erdélyi *et al.*, *Higher Transcendental Functions*, Vol. 2 (McGraw–Hill, New York, 1953).